A Method for Accurate Design of a Broad-Band Multibranch Waveguide Coupler*

K. G. PATTERSON†

Summary—A new approach is made to the problem of tapering the branch impedances for broad-band performance. A taper is proposed, which, for a 3-db branch coupler, is shown to give much better results in theory and practice than the currently used binomial taper.

Also, simple expressions are developed which enable the effects of waveguide junction discontinuities to be adequately corrected, thus allowing a greater accuracy in design to be achieved than was hitherto possible.

List of Symbols

λg_0 = Guide wavelength at the midband frequency.
f_0 = Midband frequency of a coupler.
A_k = Normalized voltage vector (k = 1, 2, 3 or 4).
Γ = Reflection coefficient.
T = Transmission coefficient.
Z_n = nth branch guide impedance.
Z_n' = Corrected nth branch guide impedance.
$z_n = Z_n/Z_0$, where Z_0 = main guide impedance.
$z_n' = Z_n'/Z_0$.
$y_n = 1/z_n$.
g_n = nth element of normalized prototype network in farads or henries with respect to unity impedance level.
ρ, σ = Constants.
N = Total number of branches.
A, B, C, D = Matrix parameters.
j = Complex operator.
a = Broad dimension of all guides.
b_0 = Narrow dimension of main and auxiliary guides.
b_n = Narrow dimension of nth branch guide.
jB = Equivalent susceptance.
m^2 = Ideal transformer impedance ratio.
θ = Electrical line length in radians.
d_n, l_n = Displacement of reference planes.
L_n = Physical length of a branch guide.
D_{np} = Spacing between branch guide centers ($p = n \pm 1$).
λ = Free-space wavelength at a frequency f.
f = Frequency.
$\omega = 2\pi f$.
$\omega_0 = 2\pi f_0$.

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† Communication from the Staff of the Research Laboratories of the General Electric Co., Ltd., Wembly, England.

Introduction

Since 1945, a considerable number of papers have been written on the subject of directional couplers, notably by Riblet [1], Mumford [2], and Lippmann [3]; much of the theory presented therein is applicable to multi-element couplers employing branch lines or guides as coupling elements. These authors, however, were principally concerned with methods of analysis of the general multi-element coupler with ideal coupling elements; when practical design information on branch couplers is required, the engineer usually turns to the report by Harrison [5].

The section of Harrison's report [5] which deals with branch couplers is based on a report by Lippmann [4] which is not generally available, and which is restricted in its scope. The essential problem of optimum taper of branch impedances is given scant consideration, and the problem of correction for the discontinuity effects at the T-junctions is not accurately treated except for certain waveguide sizes where Lippmann's results [4] are quoted.

This paper is concerned with these two problems and is intended to fill in the gaps in the present design techniques with the presentation of some simple and generally applicable design equations.

Taper of Branch Impedances

The branch waveguide coupler comprises a main guide coupled to an auxiliary guide of equal size by a number of branch guides (Fig. 1). The branch guides are usually series-connected by T-junctions to the main and auxiliary guides; their narrow dimension may be varied

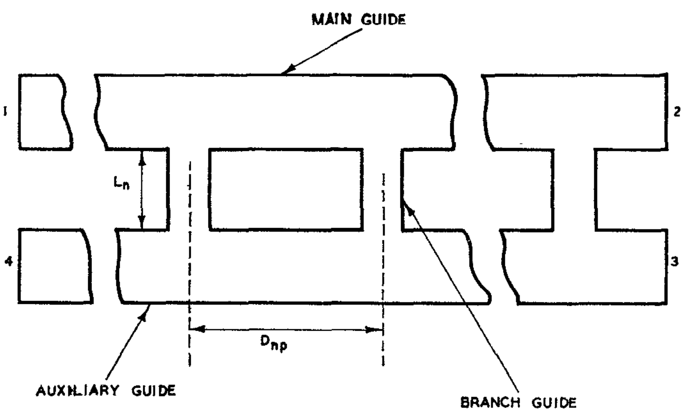

Fig. 1—Multibranch waveguide coupler, cross section.

according to the taper and coupling required; their broad dimension is made equal to that of the main and auxiliary guides. If the T-junctions are ideal; *i.e.*, there are no waveguide discontinuity effects, the length of a branch guide will be $\lambda g_0/4$, and the spacing between the corresponding points of adjacent guides will be $\lambda g/4$ at the midband or design frequency. When the ratio of input power to power coupled to the auxiliary guide is 3 db, such a coupler has all the properties of a hybrid junction.

There is a plane of symmetry between the main and auxiliary guides which bisects the branch guides transversely. Hence this type of coupler can be fully analyzed by the mode superposition method employed by Reed and Wheeler [7] and Young [8]. The substance of this method is that two related two-port circuits are derived from the coupler by first placing a magnetic and then an electric wall in the above mentioned plane of symmetry. One two-port circuit comprises a number of open-circuit stubs, and the other a number of short-circuit stubs, connected to the main or auxiliary guides (Fig. 2). At the midband frequency, these branch stubs will be $\lambda g_0/8$ long and spaced by $\lambda g_0/4$ between center lines. The performance of the coupler can now be expressed in terms of the properties of these derived two-port circuits. If the input to port 1 of the coupler is a voltage vector of unit amplitude, the emergent voltage vectors from the four ports of the coupler with each port properly matched are given by:

$$A_1 = \tfrac{1}{2}[\Gamma_- + \Gamma_+], \quad (1)$$

$$A_2 = \tfrac{1}{2}[T_- + T_+], \quad (2)$$

$$A_3 = \tfrac{1}{2}[T_- - T_+], \quad (3)$$

$$A_4 = \tfrac{1}{2}[\Gamma_- - \Gamma_+], \quad (4)$$

where the suffix $-$ refers to the short-circuit stub circuit, and the suffix $+$ refers to the open-circuit stub circuit.

Thus far, the development follows that of Reed and Wheeler [7]. At this point, the argument leading to the selection of a suitable taper can be taken up.

Each two-port circuit can be replaced by an equivalent network of lumped elements. The $\lambda g_0/8$ stubs can be simulated by the appropriate capacitance or inductance over small percentage bandwidths, as shown in the Appendix. The $\lambda g_0/4$ lengths between stubs act as impedance inverters; for simplification these will be assumed invariant with frequency. (This assumption is discussed later.) Hence the lumped networks of Fig. 3(a) and 3(b) are obtained, which are equivalent to the two-port circuits of Fig. 2(a) and 2(b) respectively, for N odd. An additional shunt element would be required for Fig. 3(a) and 3(b) for N even.

The networks of Fig. 3(a) and 3(b) are, respectively, high-pass and low-pass ladder networks. The elements of each network may be tapered according to well-known network theory [10] by the same proportionality, so that the insertion loss characteristic of each network is monotonic; *i.e.*, may be represented by a function of the form $1+x^{2N}$, where x is a frequency variable. Then, for small values of insertion loss, the magnitude of the input reflection coefficient will approximate closely to a monotonic characteristic of the form x^N. Thus for the low-pass network, $|\Gamma_-| \propto \omega'^N$, where ω' is given by (47), and for the high-pass network, $|\Gamma_+| \propto 1/\omega'^N$. The values of Γ_- and Γ_+ at f_0 for the networks equivalent to the coupler with branch impedance values given by (26) can be calculated from (14) and (21) to (23). They are

$$(\Gamma_-)_{f_0} = 0.0006(1 + j1)$$

$$(\Gamma_+)_{f_0} = 0.0006(1 - j1). \quad (5)$$

The criterion of coupler performance to be studied here is the directivity, which may be expressed as $|A_3/A_4|$. $|A_3|$ varies by <10 per cent over the 25 per cent frequency band of interest. This variation will not significantly affect the directivity. A_4 is given by (4) as the vector difference of Γ_- and Γ_+. Therefore:

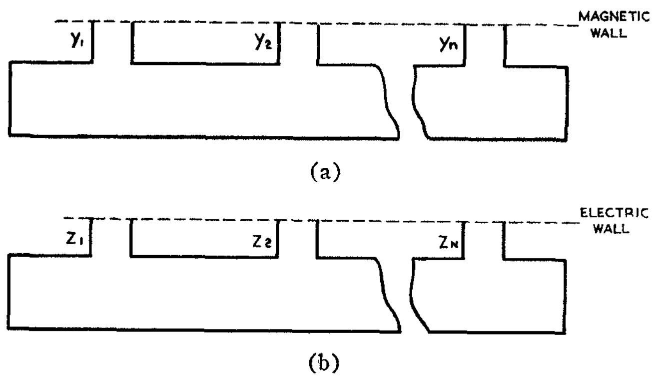

Fig. 2—Derived two-port circuits, cross sections.

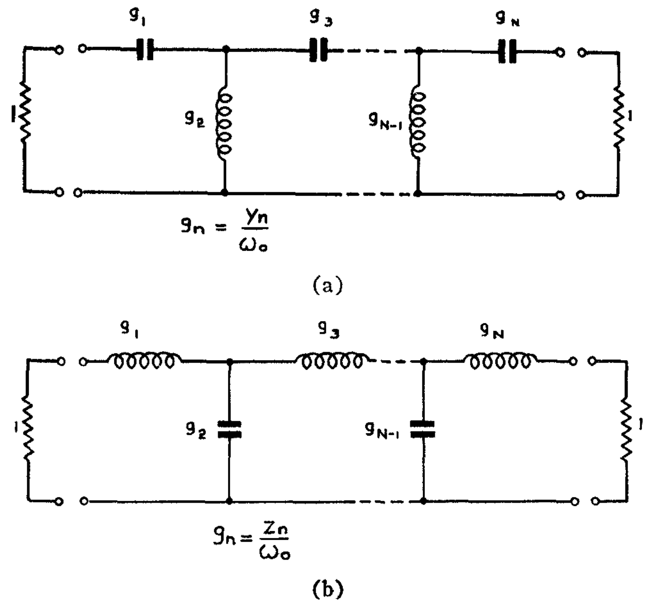

Fig. 3—Lumped element equivalent circuits.

$$|A_4| \geq \tfrac{1}{2}(|\Gamma_-| + |\Gamma_+|). \tag{6}$$

$|\Gamma_-|$ and $|\Gamma_+|$ can be written as:

$$|\Gamma_-| = |\Gamma_-|_{f_0}\left(\frac{\omega'}{\omega_0}\right)^N; \quad |\Gamma_+| = |\Gamma_+|_{f_0}\left(\frac{\omega_0}{\omega'}\right)^n, \tag{7}$$

where $|\Gamma_-|_{f_0} = |\Gamma_+|_{f_0} = 0.00085$, and $N = 5$.

Hence, (6) may be written as

$$|A_4| \leq 0.00042\left[\left(\frac{\omega'}{\omega_0}\right)^N + \left(\frac{\omega_0}{\omega'}\right)^N\right]. \tag{8}$$

For

$$\frac{f - f_0}{f_0} = 0.05, \qquad \frac{\omega'}{\omega_0} = 1.16, \qquad |A_4| \leq 0.0011$$

and directivity

$$\left|\frac{A_3}{A_4}\right| \geq 56 \text{ db for } \left|\frac{f - f_0}{f_0}\right| \leq 0.05.$$

These figures suggest that the so-called "maximally flat" taper when applied to 3-db branch couplers should give good results over the small frequency range for which the approximations are valid.

This taper is given by well-known network theory for the high-pass ladder network of Fig. 3(a), as

$$\frac{1}{g_n} = \rho \sin (2n - 1)\frac{\pi}{2N}, \tag{9}$$

and for the low-pass ladder network of Fig. 3(b), as

$$g_n = \rho \sin (2n - 1)\frac{\pi}{2N}, \tag{10}$$

where ρ is a constant determined by the equivalence of the lumped element networks to the waveguide circuits of Fig. 2. From either (9) or (10), the taper of branch impedances of the coupler is given by

$$Z_n = \sigma \sin (2n - 1)\frac{\pi}{2N}, \tag{11}$$

where σ is a constant determined by the coupling ratio of the branch coupler.

The main restriction to this argument lies in the assumption that the $\lambda g_0/4$ lengths between branches are frequency invariant. The results obtained, however, using the sine taper of (11) for 3-db couplers, suggest that the approximations here are not worse than those made by classical binomial theory [12] of weak coupling and frequency invariant coupling elements. However, it is not expected that the approximations made in the foregoing analysis should be valid outside the range

$$\left|\frac{\lambda g - \lambda g_0}{\lambda g_0}\right| < 0.1,$$

and it must be presumed that the wide useful range of

$$\left|\frac{\lambda g - \lambda g_0}{\lambda g_0}\right| < 0.2$$

obtained in the present instance with a sine taper is due to extra compensating effects not yet explained.

The theoretical characteristics of Figs. 4–6 were calculated with the aid of an electronic computer, following Reed and Wheeler's [7] method of analysis. Once the branch sizes are known—the analysis is straightforward, and there is no necessity to assume that branch length or branch spacing are frequency invariant. There is therefore no restriction in the theoretical calculations on that score, but the assumption is made that the T-junctions are ideal; *i.e.*, without reactive discontinuities. However, as is shown in the succeeding sections of this paper, the effects of these discontinuities can be largely corrected in a practical design, so that measured characteristics should agree fairly well with the theoretical ones. The measured characteristics shown in Fig. 4 are of a 3-db coupler designed by the methods of this paper, but there the extent of the correction for discontinuity effects has been limited by the physical requirement for equal branch lengths. The agreement between measured and theoretical performance in this instance is only fair.

Fig. 5 shows the theoretical performance of a five-branch 3-db coupler designed by Harrison's method [5] with binomially tapered voltage coupling coefficients, the resultant branch impedance taper being somewhat "sharper" than binomial. The theoretical performance of a 3-db coupler with binomial taper of the branch impedances, actual branch size being found by the method given later in this paper, is shown in Fig. 6. It will be seen that the input match has improved over that of Fig. 5. This is part of a general trend of improvement that takes place when the binomial taper is "flattened" towards the values given by the sine taper of (11). An important fact is that the theoretical directivity is not infinite at midband for any of the tapers discussed here, least of all for the binomial taper of Fig. 5. This is not an error in calculation, and may be easily checked by using (19) and (21)–(23). This deficiency is avoided in the design method employed by Reed and Wheeler [7] and Young [8], which has been further developed by Reed [9]. In this method, two branch impedance values are determined by applying the two essential midband conditions for coupling and for directivity. Thus the midband performance is assured, but no broad-band condition is applied.

It may be noted from Figs. 4–6 that the theoretical coupling characteristic shown as power division ratio, $|A_3/A_2|^2$, is not much affected by these changes of taper.

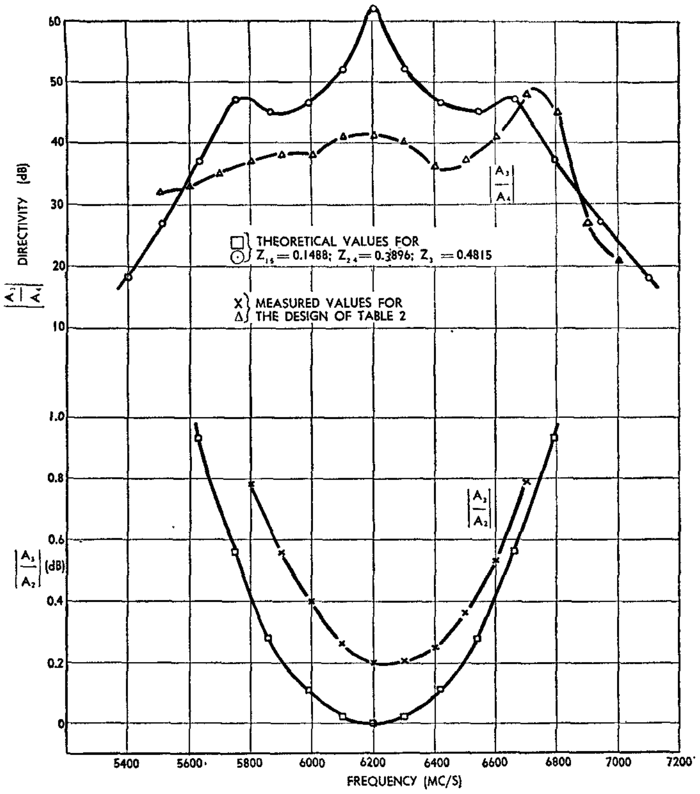

Fig. 4—Characteristics of branch coupler with sine taper.

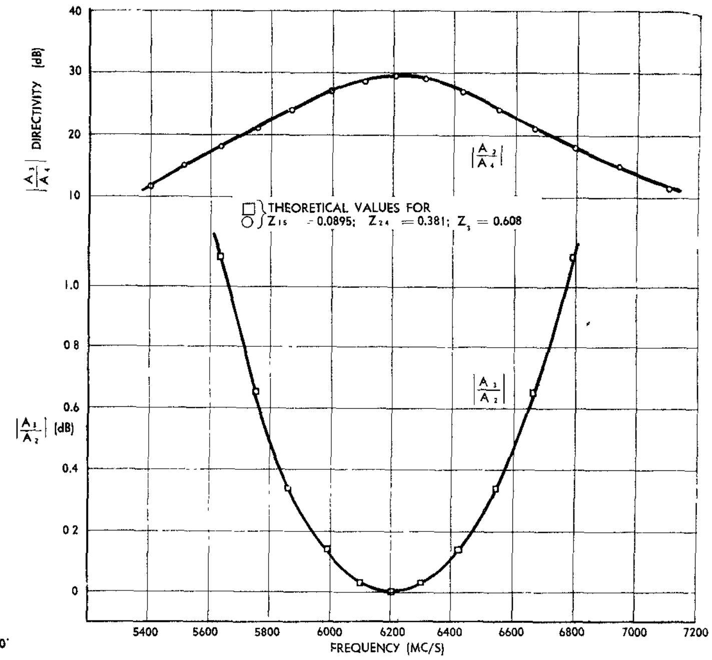

Fig. 5—Characteristics of branch coupler with binomial taper of voltage coupling coefficients.

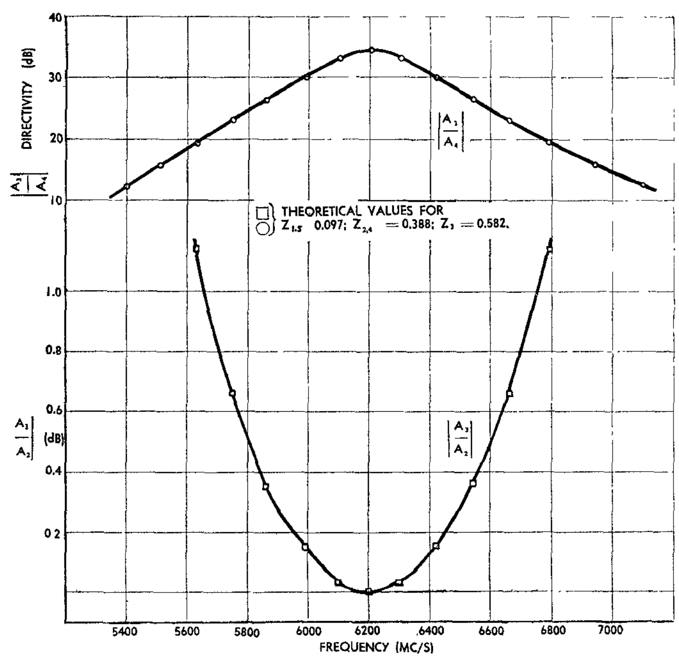

Fig. 6—Characteristics of branch coupler with binomial taper of branch impedances.

Exact Branch Impedance

The design methods of Harrison [5] and Crompton [6] consider the multi-element coupler as a cascaded series of two-element couplers. The coupling factor of the multi-element coupler is then found by a simple summation of the power coupled by the two-element couplers. This method cannot easily be applied to other than binomial couplers and is strictly applicable only if the total coupling is weak.

A more direct method, which avoids the above restrictions and is generally applicable, is possible. It is based on Reed and Wheeler's analysis, which can be developed in the following manner.

The transfer matrix of a lossless two-port circuit may be written as

$$\begin{bmatrix} A & jB \\ jC & D \end{bmatrix}, \tag{12}$$

taking Fig. 9 as a definition, so that the transmission and reflection coefficients of this circuit can be expressed as:

$$T = \frac{2}{A + jB + jC + D}, \tag{13}$$

$$\Gamma = \frac{A + jB - jC - D}{A + jB + jC + D}. \tag{14}$$

Let the matrix (12) apply in particular to the circuit of Fig. 2(a) with open-circuit stubs, at the midband frequency. Then the corresponding matrix of the circuit of Fig. 2(b) with short-circuit stubs can be shown to be

$$\begin{bmatrix} A & -jB \\ -jC & D \end{bmatrix}, \tag{15}$$

when there are an odd number of stubs. With an even number of stubs, both A and D change sign; the expressions for A_k are the same as for the odd-number case except for an interchange of A_2 and A_3.

The tapers considered here yield a circuit which is symmetrical end for end; hence,

$$A = D$$

and (1)–(4) can be developed as

$$A_1 = \frac{1}{R^2}(B^2 - C^2), \tag{16}$$

$$A_2 = \frac{2}{R^2}(A + D), \tag{17}$$

$$A_3 = \frac{2j}{R^2}(B - C), \tag{18}$$

$$A_4 = \frac{-j}{R^2}(A + D)(B + C);. \tag{19}$$

where

$$R^2 = (A + D)^2 + (B + C)^2.$$

The power balance at midband can now be expressed as

$$\left|\frac{A_3}{A_2}\right|^2 = \left|\frac{B + C}{A + D}\right|^2, \tag{20}$$

for the coupler with an odd number of branches, and as the reciprocal of this when there are an even number of branches.

As a specific example, the foregoing ideas will be applied to the design of a five-branch 3-db coupler.

Let the normalized branch impedances be, in order,

$$z_1 \quad z_2 \quad z_3 \quad z_2 \quad z_1.$$

When the component matrices are multiplied out, the elements of the matrix (12) are found to be, at f_0:

$$A = D = (z_1 z_2 - 1)(z_2 z_3 - 2) - 1, \tag{21}$$

$$B = (z_1 z_2 - 1)(2z_1 + z_3 - z_1 z_2 z_3), \tag{22}$$

$$C = z_2(z_2 z_3 - 2). \tag{23}$$

When the impedance taper is known, the matrix elements can be expressed in terms of z_1, or any one-branch impedance. The sine taper given by (11) applied to five branches gives

$$z_2 = 2.618 z_1 \qquad z_3 = 3.236 z_1. \tag{24}$$

Hence,

$$\left|\frac{A_3}{A_2}\right| = \left| z_1 \cdot \frac{5.236 - 22.18 z_1^2 + 11.09 z_1^4}{1.000 - 13.71 z_1^2 + 22.18 z_1^4} \right|. \tag{25}$$

For a 3-db coupler:

$$\left|\frac{A_3}{A_2}\right| = 1.000,$$

and (25) can be solved for z_1. It is not a difficult equation to solve, bearing in mind that only the smallest root (of the order of 0.1) is required. The method of iterative approximation is adequate. The resultant values for z_n are

$$\left.\begin{aligned} z_1 &= 0.1488 \\ z_2 &= 0.3896 \\ z_3 &= 0.4815 \end{aligned}\right\}. \tag{26}$$

These values will need to be corrected for the effect of the T-junction discontinuities. The necessary correction is given by (33), which can be stated as

$$z_n' = \frac{z_n}{m^2 \sin\theta}, \tag{27}$$

where z_n' is the corrected value and m^2, θ are found as shown later.

The narrow dimension b_n of the nth branch guide is found from the relation:

$$z_n' = \frac{b_n}{b_0}, \tag{28}$$

where b_0 is the narrow dimension of main and auxiliary guides.

Exact Branch Length

The existance of reactive discontinuity effects at the junction of a branch guide with main or auxiliary guides considerably modifies the performance of the branch guide. The effective size of a branch guide connected between two open E-plane junctions, which are commonly used in branch couplers, may be derived as follows.

Following Marcuvitz [11], the E-plane junction of Fig. 7(a) may be represented by the lumped equivalent network of Fig. 7(b), with suitable choice of reference planes T_1 and T_2. Values are given by Marcuvitz [11] for all the necessary parameters, but it has been pointed out to this author that the family parameter $b/\lambda g$ occurring inside the graphs of all Marcuvitz' E-plane curves should be altered to $2b/\lambda g$.

Let a branch guide of surge impedance Z_n' be connected between two such junctions and let the electrical length of the branch guide measured between the corresponding terminal planes T_1 of the junctions be θ radians. The lumped equivalent circuit of a branch guide can now be drawn, including the equivalent parameters of the junctions, the susceptances jB and ideal transformers of impedance ratio m^2, as shown in Fig. 8.

Defining the transfer matrix as in Fig. 9, the transfer matrix M_1 of the network of Fig. 8, assuming lossless components, is given by

$$M_1 = \begin{bmatrix} \cos\theta - BZ_n''\sin\theta & jZ_n''\sin\theta \\ j\left[2B\cos\theta + \left(\frac{1}{Z_n''} - B^2Z_n''\right)\sin\theta\right] & \cos\theta - BZ_n''\sin\theta \end{bmatrix}, \quad (29)$$

writing

$$Z_n'' \equiv m^2 Z_n'. \quad (30)$$

With discontinuity effects the branch guide would be of length $\lambda g_0/4$ and of surge impedance Z_n, for instance. The transfer matrix M_2 of the equivalent line network is

$$M_2 = \begin{bmatrix} 0 & jZ_n \\ \frac{j}{Z_n} & 0 \end{bmatrix}. \quad (31)$$

The matrices M_1 and M_2 become identical if

$$\cos\theta = BZ_n'' \quad (32)$$

and

$$Z_n'' = \frac{Z_n}{\sin\theta}. \quad (33)$$

This follows by equating A and B elements and substituting these results in the C elements.

Initially only Z_n, the surge impedance of a branch guide between ideal junctions, is known. However, the actual values required, Z_n' and θ, can be found quite easily by successivè approximation using (30), (32), and (33). One or two iterations are usually sufficient.

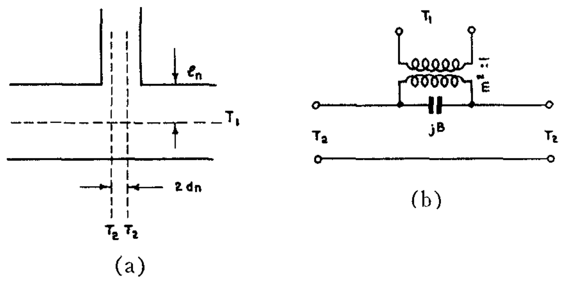

Fig. 7—Open E-plane waveguide junction (cross section), and lumped equivalent circuit.

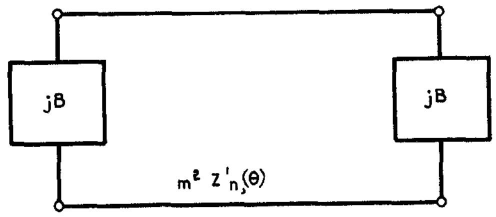

Fig. 8—Equivalent branch circuit.

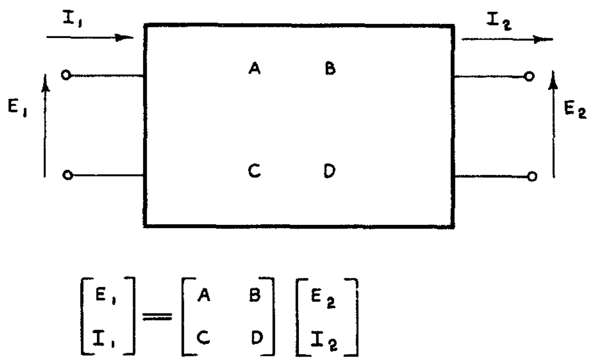

Fig. 9—Transfer matrix definition.

The exact length, L_n (Fig. 1), of a branch guide is given by

$$L_n = \frac{\lambda g_0}{2\pi}\theta_n - 2l_n, \quad (34)$$

where l_n is the displacement of the reference plane T_1 (Fig. 7).

The author has found that the lengths obtained by these means for the larger branch guides of a five-branch 3-db coupler are almost exact. The approximate expression given by Harrison [5] for the quantity L_n, *viz.*,

$$L_n = \frac{\lambda g_0}{4} - \frac{2b_n}{\pi}\left(1 + \log_e \frac{b_0}{2b_n}\right), \quad (35)$$

has been found to predict values which are 6 per cent short.

Branch Spacing

Little manipulation is required to obtain this quantity. The spacing D_{np} between center lines of adjacent branch guides is given by

$$D_{np} = \frac{\lambda g_0}{4} + d_n + d_p, \tag{36}$$

where $p = n \pm 1$ and $2d_n$ is the distance between the reference planes T_2 (Fig. 7), of the nth branch guide junction.

Comparison of Theoretical and Practical Design

A five-branch coupler was designed by the foregoing methods with main and auxiliary guide dimensions of 1.372 inches×0.622 inch. The branch impedances followed the sine taper of (11). For comparison with a coupler actually constructed, the coupling loss at the midband frequency of 6200 mc has been made 2.90 db; *i.e.*, the power division ratio $|A_3/A_2|^2 = 0.22$ db. This necessitates a recalculation of the values for z_n given by (26). These become

$$\left.\begin{aligned} z_1 &= 0.1513 \\ z_2 &= 0.3958 \\ z_3 &= 0.4892 \end{aligned}\right\}. \tag{37}$$

The final calculated dimensions are given in Table I.

TABLE I

($\lambda g_0/4 = 0.661$ inch)

n	z_n'	L_n (inches)	D_{np} (inches)
1, 5	0.156	0.572	0.669
2, 4	0.435	0.499	0.679
3	0.556	0.484	—

In order to simplify the mechanical design, it is common practice to make all branch arms the same length, choosing a power-weighted average of the correct branch lengths; *i.e.*,

$$\frac{\sum z_n{}^2 L_n}{\sum z_n{}^2}.$$

For the design of Table I, this common length would be about 0.496 inch. In an actual five-branch coupler with the dimensions given in Table II designed with a sine taper of branch impedances, the branch length required to center the power division characteristic at 6200 mc was found to be 0.496 inch. The agreement is fortuitous; an error of at least 1 per cent might be expected. The measured power division ratio, $|A_3/A_2|^2$, was 0.2 db at midband, 6200 mc.

The practical values of branch impedance should be related to the theoretical values of (37) by the previously used relation (33), restated as

$$z_n' = \frac{z_n}{\sin 2\pi L_n/\lambda g_0}, \tag{38}$$

where L_n is given the same value for each branch of 0.496 inch, and z_n is given by (37). Hence at the midband frequency of 6200 mc:

$$z_n' = \frac{z_n}{0.924}. \tag{39}$$

This is a short cut on the design (27), the parameter m^2 having been eliminated together with its associated line length l_n.

TABLE II

($\lambda g_0/4 = 0.661$ inch)

n	z_n'	$z_n/0.924$	L_n (inches)	D_{np} (inches)
1, 5	0.165	0.163	0.496	0.668
2, 4	0.436	0.428	0.496	0.679
3	0.536	0.529	0.496	—

The actual value of z_n' required differs by 2 per cent or less from the value predicted by (39).

Fig. 4 shows the directivity and coupling characteristics of a coupler made to the dimensions given in Table II and measured with all output ports terminated by loads with less than 0.005 reflection coefficient.

With incomplete correction of junction discontinuity effects, due to using a common branch length, the measured performance would not be expected to equal the theoretical performance. Nevertheless, it is considerably better than the theoretical performance of the binomial couplers.

Conclusion

A considerable improvement, which is particularly applicable to 3-db couplers, can easily be made on the currently used binomial design of branch couplers.

The practical values of Table II differ from the predicted values by <2 per cent in branch length and about 2 per cent in branch impedance. This is equivalent to <1 per cent difference in midband frequency, and about 0.3 db in power division ratio, $|A_3/A_2|^2$, at midband.

Appendix

Lumped Element Equivalence

The normalized input reactance z_i of a short-circuit waveguide stub may be expressed as

$$z_i = z_n \tan \theta/2.$$

Differentiating with respect to frequency f,

$$\frac{dz_i}{df} = \frac{z_n}{2} \sec^2 - /2 \cdot \frac{d\theta}{d\lambda g} \cdot \frac{d\lambda g}{d\lambda} \cdot \frac{d\lambda}{df}, \tag{40}$$

and since

$$\frac{d\theta}{d\lambda g} = -\frac{\theta}{\lambda g}, \tag{41}$$

$$\frac{d\lambda g}{d\lambda} = \left(\frac{\lambda g}{\lambda}\right)^3, \tag{42}$$

$$\frac{d\lambda}{df} = -\frac{\lambda}{f}, \tag{43}$$

then

$$\frac{dz_i}{df} = \frac{z_n}{2}\sec^2\theta/2\cdot\frac{\theta}{f}\cdot\left(\frac{\lambda g}{\lambda}\right)^2. \tag{44}$$

When $f=f_0$ (6200 mc),

$$\left(\frac{\lambda g}{\lambda}\right)^2 \approx 2 \quad \text{and} \quad \theta = \frac{\pi}{2},$$

$$\left(\frac{dz_i}{df}\right)_{f=f_0} = \frac{\pi z_n}{f}. \tag{45}$$

Thus, to a first-order approximation valid over a small frequency range around f_0, the slope of z_i is linear and positive with frequency, and hence the short-circuit stub can be represented as an inductance L_n, in conjunction with a certain frequency variable, ω', given by

$$L_n = \frac{z_n}{\omega} \text{ (henries, with respect to unity impedance level)}, \tag{46}$$

$$\omega' = \omega_0\left(1 + \pi\frac{\omega - \omega_0}{\omega_0}\right). \tag{47}$$

A similar argument applies to the open-circuit stubs, the slope of the input admittance being linear and positive with frequency to a first-order approximation so that the open-circuit stub can be represented as a capacity C_n, in conjunction with the frequency transformation of (47). C_n is given by

$$C_n = \frac{y_n}{\omega_0} \text{ (farads, with respect to unity impedance level)}. \tag{48}$$

Bibliography

[1] H. J. Riblet, "A mathematical theory of directional couplers," Proc. IRE, vol. 35, pp. 1307–1313; November, 1947.
[2] W. W. Mumford, "Directional couplers," Proc. IRE, vol. 35, pp. 160–165; February, 1947.
[3] B. A. Lippmann, "Theory of Directional Couplers," Mass. Inst. Tech. Radiation Lab. Rept. 860; 1945.
[4] B. A. Lippman, "Equivalent Circuit Analysis of Directional Couplers," Mass. Inst. Tech. Radiation Lab. Rept. 41; March 1, 1945.
[5] R. J. Harrison, "Design Considerations for Directional Couplers," Mass. Inst. Tech. Radiation Lab. Rept. 724; 1945.
[6] J. W. Crompton, "A contribution to the design of multi-element directional couplers," *Procl IEE (London)*, vol. 104, pt. C, pp. 398–402; 1957. (Monograph No. 230R.)
[7] J. Reed, and G. J. Wheeler, "A method of analysis of symmetrical four port networks," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 246–252; October, 1956.
[8] L. Young, "Branch guide couplers," *Proc. NEC*, vol. 12, pp. 723–732; 1956.
[9] J. Reed, "The multiple branch waveguide coupler," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 398–403; October, 1958.
[10] S. B. Cohn, "Direct coupled resonator filters," Proc. IRE, vol. 45, pp. 187–196, February, 1957.
[11] N. Marcuvitz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., p. 339; 1948.
[12] A. T. Starr, "Radio and Radar Technique," Sir Isaac Pitman and Sons, London, Eng., p. 163; 1953.

Correspondence

The Analogy between the Weissfloch Transformer and the Ideal Attenuator (Reflection Coefficient Transformer) and an Extension to Include the General Lossy Two-Port*

Weissfloch's transformer theorem states that at certain pairs of reference planes a lossless two-port can be represented by an ideal transformer. There are many proofs of this important theorem. One of the most interesting is due to Bolinder,[1] who uses properties of the bilinear transformation. For lossless two-ports, the transformations will belong to the Fuchsian[2] group. This means that the isometric[1,2] circles are orthogonal to the principal circle. In the reflection coefficient plane (where Bolinder proves the theorem), the principal circle is the unit circle. The fixed points of the transformation will be on the unit circle or in a pair inverse with respect to the unit circle. Bolinder then uses lengths of lossless line to move the fixed points to the positions $\Gamma = \pm 1$. In the impedance plane this corresponds to fixed points of 0 and ∞. Therefore the transformation can be written as $Z' = k^2 Z$ and the transformer theorem is proven. The transformation through the two-port at any pair of reference planes, in either the reflection coefficient or the impedance plane, can be done by inversion in the isometric circles and a reflection in the line of symmetry,[3] as described by Ford[2] and Bolinder.[1]

The reflection coefficient transformer-(ideal attenuator), described by Altschulter and Kahn,[4] has a scattering matrix

$$S = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

where K is a real number and, for an attenuator, less than unity. The transformation in the reflection coefficient plane is $\Gamma' = K^2\Gamma$, while in the impedance plane the corresponding relation is

$$Z' = \frac{\dfrac{Z(1+K^2)}{2K} + \dfrac{(1-K^2)}{2K}}{\dfrac{Z(1-K^2)}{2K} + \dfrac{1+K^2}{2K}}.$$

The fixed points of this transformation are ± 1 in the impedance plane and 0 and ∞ in the reflection coefficient plane. Both transformers produce hyperbolic transfor-

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[1] E. F. Bolinder, "Impedance and polarization-ratio transformations by a graphical method using the isometric circles," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 176–180; July, 1956.

[2] L. R. Ford, "Automorphic Functions," 2nd ed., Chelsea Publishing Co., New York, N. Y.; 1951.

[3] If the transformation is loxodromic, a rotation must be added.

[4] H. M. Altschulter and W. K. Kahn, "Nonreciprocal two-ports represented by modified Wheeler networks," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 228–233; October, 1956.